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Abstract

In this paper, we study a class of stochastic partial differential equations (SPDEs) driven by space-time fractional noises. Our method consists in studying first the nonlocal SPDEs and showing then the convergence of the family of these equations and the limit gives the solution to the SPDE.

Key Words: Stochastic Partial Differential Equation, Space-time Fractional Noise.
MSC: 60H15

1 Introduction

Recently, stochastic partial differential equations are studied mainly as alternative physical models for some complex and chaotic natural phenomena. For example, in the study of turbulent phenomena, a reduction must be made although the Navier-Stokes equations are believed to catch the motions of all different sorts of flows of incompressible fluids. It is thus hopeless, at least under the current technologies and the current computational power (it might be changed with the arrival of quantum computers which is still in the remote future), to understand the solutions to the Navier-Stokes equations subject to complicated boundary conditions with precision and mathematical rigor. One of the ideas in the fluid dynamics is to combine the equations of motions with statistical ideas. Statistical fluid mechanics has been the main tool for the understanding of the turbulent flows. Traditional statistical fluid mechanics is based on the hypothesis (which has not been proved yet) that there is an underlying invariant measure with respect to the non-linear semigroup defined by the Navier-Stokes equations, and is not based on the use of stochastic evolution models. Itô's calculus and its generalizations to infinite dimensional state spaces such as Malliavin Calculus etc., on the other hand, provide the possibility to construct useful stochastic evolution models directly. A class of simple models can be constructed by simplifying the equations of motions and enhanced by adding suitable noise terms in order to recover essential features in the original physical laws. For example, for the equation of motion for an incompressible fluid:

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = \nu \Delta u - \nabla p \text{ and } \nabla \cdot p = 0$$

where u describes the velocity of the flow and p is the pressure. In order to apply the familiar theory of parabolic equations, a simple way to make reduction is to drop the pressure term ∇p from the first equation, so that the Navier-Stokes equations become a parabolic system

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = \nu \Delta u$$

which preserves the non-linear convection, but certainly many interesting features are lost. To recover the chaotic nature of the fluids, we may add a noise term to the parabolic equation, which thus leads to the following stochastic partial differential equations

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = \nu \Delta u + W$$

where W should be modeled by a space-time random field, and W can be considered as a kind of radom perturbations, or as external random force applied to the fluid in question. This kind of stochastic partial differential equations has received study in the recent years, and a lot of interesting results have been obtained. While, there is no certain rules which dictate the choice of a noise term, and the choice of a reasonable stochastic process really depends on the equation of motion in question, by taking into account of the physical meaning as far as possible.

In this paper, we study the following stochastic partial differential equation (SPDE):

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + g(u(t,x), \frac{\partial}{\partial x}u(t,x)) + \dot{B}^H, \\ u(t,0) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.1}$$

where $(t,x) \in [0,T] \times D$, $D = [0,\infty)$, and

$$\dot{B}^H = B^H(dt,dx) = \frac{\partial^2 B^H(t,x)}{\partial t \partial x}$$

is a space-time fractional noise with Hurst parameter $H = (h_1, h_2)$, $h_i \in (0,1)$ for $i = 1,2$, see below for a precise definition.

SPDEs driven by Gaussian noises have been widely studied where the non-linear term g depends only on u, see for example Walsh [20], Da Prato-Zabczyk [5], Hu et al. [10] and Fuhrman et al. [6]. These theories and their applications are now classic and mature.

If g has a particular form depending on u and $\frac{\partial}{\partial x}u$ as well, such as the non-linear term in the Burgers equation, the SPDE has been considered by many authors. Mohammed and Zhang [15] studied the dynamics of the stochastic Burgers equation on the unit interval driven by affine linear noise. Using multiplicative ergodic theory techniques, they established the existence of a discrete non-random Lyapunov spectrum for the cocycle. They also proved an existence theorem for solutions of the stochastic Burgers equation on the unit interval subject to the Dirichlet boundary condition and the anticipating initial velocities in [16]. Wang et al. [21] proposed an L^2-gradient estimate for the corresponding Galerkin approximations, and the log-Harnack inequality was established for the semigroup associated to a class of stochastic Burgers equations. Hairer and Voss [8] discussed the numerical methods of various finite-difference approximations to the stochastic Burgers equation.

Recently, a class of special Gaussian processes called fractional Brownian motion (fBm) has been attracted attention due to their useful feature of preserving long term memory, and a large number of interesting results from scaling invariance to the description of their laws as random fields have been established by various authors. The study of these Gaussian processes has its historical motivation from their applications in hydrology and telecommunication, and have been applied to the mathematical finance, biotechnology and biophysics, see for example [19, 7, 12] and the literature therein. Coutin and Qian [4], Mandelbrot and Van Ness [13] and some other authors have proposed a theory of stochastic calculus for a class of continuous stochastic processes with long time memory, including the fractional Brownian motions as arachtypical examples. Neuenkirch and Tindel [17] studied the least square-type estimator for an unknown parameter in the drift coefficient of a stochastic differential equation with additive fractional noise modeled by a fractional Brownian motion with Hurst parameter $H > 1/2$. Balan [1] identified necessary and sufficient conditions for the existence of a random field solution for some linear stochastic partial differential equations of parabolic and hyperbolic type. While, Bo et al. [3] considered stochastic Cahn-Hilliard equations with fractional noises, the existence, uniqueness and regularity of the solutions were obtained. In [11] the stochastic Burgers equation driven by the fractional noise was studied, a global mild solution was obtained and the existence of a distribution density of the solution was also established.

The goal of this paper is to study SPDE (1.1) where g is a function depending on both u and $\frac{\partial}{\partial x}u$, and where

$$\dot{B}^H = B^H(dt, dx) = \frac{\partial^2 B^H(t,x)}{\partial t \partial x}$$

is a space-time fractional noise. We study the existence and uniqueness of solution to this class of SPDEs, and the regularity of its solution.

There are two key steps in the present approach. The first step is to study the following non-local SPDE:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + g(u(t,x), u^\theta(t,x)) + \dot{B}^H, \\ u^\theta(t,x) = \frac{1}{\theta}\left(u(t,x+\theta) - u(t,x)\right), \\ u(t,0) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{1.2}$$

where $\theta \in \mathbb{R}$, $\theta \neq 0$, is a parameter. Note that u^θ is subject to the same boundary condition as that of u. For each $\theta \neq 0$, the unique solution to the SPDE above depends on θ, and thus is denoted by $u(t,x,\theta)$. In the second step, we show that the family of solutions $u(t,x,\theta)$ converges in an appropriate function space to a limit $u(t,x)$ as $\theta \to 0$, which provides the solution of SPDE (1.1). A space-time fractional noise is a two-parameter Gaussian random field which can be defined similarly as in one parameter case, and can be specified in terms of its covariance function. A few definitions about the stochastic integration theory for such space-time fractional noise will be recalled in the following sections. As matter of fact, the regularity properties of space-time fractional noises are fully reflected in the Hurst parameter $H = (h_1, h_2)$, and our main result of this paper can be simply described in terms of two parameters as following: if $2h_1 + h_2 > 2$ (which thus excludes the case of space-time white noise for which $h_1 = h_2 = \frac{1}{2}$), then SPDE (1.1) admits a unique solution which has nice regularity.

Hairer and Voss [8] studied a stochastic partial differential equation where g has a special form, driven by a space-time white noise. While, in our setting we allow the non-linear term which depends on both u and its space derivative $\frac{\partial}{\partial x}u$ does not possess a convenient form, thus causes essential difficulty. While our stochastic equation is driven by a space-time fractional noises which in some sense alleviates the technical difficulties.

The paper is organized as following. After introducing the stochastic integration theory for fractional noise and the class of SPDEs in the next section, we study the well solvability of non-local SPDE, including existence, uniqueness and regularity of solutions in Section 3. Section 4 is devoted to the existence and uniqueness of the SPDE. We collect several estimates about Green functions used in the main text in the Appendix.

Throughout the paper, the generic positive constant C may be different from line to line.

2 Preliminaries

In this part, we first recall a few definitions about the fractional noise and their stochastic integrals. The technical assumptions which will be enforced in the present paper are stated clearly, and some a priori estimates are established.

2.1 Fractional noise

A one-dimensional fractional Brownian motion $W^h = \{W_t^h,\ t \in [0,T]\}$ with Hurst parameter $h \in (0,1)$ on $[0,T]$ is a centered Gaussian process on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with its covariance function given by

$$\mathbb{E}\left\{W_t^h W_s^h\right\} = \frac{1}{2}\left(t^{2h} + s^{2h} - |t-s|^{2h}\right).$$

The existence of such a Gaussian process and the regularity of its sample paths are well documented. Other equivalent definitions of fractional Brownian motion and its analysis may be found in [13, 18].

Similarly, we may generalize the definition to fractional noises with two parameters (see also Jiang et al. [11] for further details).

Definition 2.1. *A one-dimensional double-parameter fractional Brownian field $B^H = \{B^H(t,x),$ $(t,x) \in [0,T] \times D\}$ with Hurst parameter $H = (h_1, h_2)$ for $h_i \in (0,1)$ and $i \in \{1,2\}$, where $D = (0,\infty)$, is a centered Gaussian field defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with covariance*

$$\begin{aligned} \mathbb{E}\left\{B^H(t,x)B^H(s,y)\right\} &= \frac{1}{4}\left(t^{2h_1} + s^{2h_1} - |t-s|^{2h_1}\right) \\ &\quad \times \left(x^{2h_2} + y^{2h_2} - |x-y|^{2h_2}\right) \\ &:= R(t,s;x,y) \end{aligned} \tag{2.1}$$

for all $t, s \in [0,T]$ and $x, y \in D$.

Let $\mathcal{E}$ denote the collection of all step functions defined on $[0,T]\times D$ and L^2_H denote the Hilbert space of the closure of $\mathcal{E}$ under scalar product

$$\langle I_{[0,t]\times[0,x]}, I_{[0,s]\times[0,y]}\rangle_{L^2_H} = R(t,s;x,y).$$

Then the mapping $I_{[0,t]\times[0,x]} \to B^H(t,x)$ can be extended to an isometry between L^2_H and the Gaussian space $\mathcal{H}$ associated with B^H.

Remark 2.2. *In this paper we only consider the one-dimensional double-parameter fractional Brownian field with Hurst parameter $H=(h_1,h_2)$, where $h_i\in(\frac{1}{2},1)$, $i=1,2$.*

Introduce the square integrable kernel

$$K_H(t,s;x,y) = c_H s^{\frac{1}{2}-h_1} y^{\frac{1}{2}-h_2} \int_s^t \int_y^x (u-s)^{h_1-\frac{3}{2}} u^{h_1-\frac{1}{2}} (z-y)^{h_2-\frac{3}{2}} z^{h_2-\frac{1}{2}} dz du$$

and its derivative

$$\frac{\partial^2}{\partial t\partial x} K_H(t,s;x,y) = c_H (t-s)^{h_1-\frac{3}{2}} \left(\frac{t}{s}\right)^{h_1-\frac{1}{2}} (x-y)^{h_2-\frac{3}{2}} \left(\frac{x}{y}\right)^{h_2-\frac{1}{2}}.$$

Define the operator K_H^* from $\mathcal{E}$ to $L^2([0,T]\times D)$ by

$$(K_H^*\phi)(s,y) = \int_s^T \int_y^\infty \phi(t,x) \frac{\partial^2}{\partial t\partial x} K_H(t,s;x,y) dt dx.$$

It is easy to check that

$$(K_H^* I_{[0,t]\times[0,x]})(s,y) = K_H(t,s;x,y) I_{[0,t]\times[0,x]}(s,y),$$

and

$$\begin{aligned} \langle K_H^* I_{[0,t]\times[0,x]}, K_H^* I_{[0,s]\times[0,y]}\rangle_{L^2([0,T]\times D)} &= R_H(t,s;x,y) \\ &= \langle I_{[0,t]\times[0,x]}, I_{[0,s]\times[0,y]}\rangle_{L^2_H}. \end{aligned}$$

Hence, the operator K_H^* is an isometry between $\mathcal{E}$ and $L^2([0,T]\times D)$ which can be extended to L^2_H. By definition

$$B(t,x) = B^H\left(K_H^{*-1}(I_{[0,t]\times[0,x]})\right), \quad (t,x)\in[0,T]\times D,$$

is a Brownian sheet, and in turn the fractional noise has a representation

$$B^H(t,x) = \int_0^t \int_0^x K_H(t,s;x,y) B(ds,dy).$$

The following embedding property enables us to define the integral for $\phi\in L^2_H$ with respect to B^H.

Proposition 2.3. *For $h>1/2$, $L^2([0,T]\times D)\subset L^{\frac{1}{h}}([0,T]\times D)\subset L^2_H$.*

The integral $\int_0^t \int_0^x \phi(s,y)B^H(ds,dy)$ is defined by

$$\int_0^t \int_0^x \phi(s,y)B^H(ds,dy) = \int_0^t \int_0^x (K_H^*\phi)(s,y)B(ds,dy). \tag{2.2}$$

For $0 \le s < t \le T$ and $x, y \in D$ define

$$\Psi_h(t,s,x,y) := 4h_1h_2(2h_1-1)(2h_2-1)|t-s|^{2h_1-2}|x-y|^{2h_2-2}.$$

A routine calculation shows the equivalence of the stochastic integrals defined in Jiang et al. [11] and those in this section for functions in L_H^2.

Proposition 2.4. *For $f, g \in L_H^2$ we have*

$$\mathbb{E}\int_0^t \int_D f(s,x)B^H(dx,ds) = 0$$

and

$$\begin{aligned}&\mathbb{E}\left\{\int_0^t \int_D f(s,x)B^H(dx,ds)\int_0^t \int_D g(s,x)B^H(dx,ds)\right\}\\ =& \int_{[0,t]^2}\int_{D^2} \Psi_h(u,v,x,y)f(u,x)g(v,y)dydxdvdu.\end{aligned}$$

In what follows, $\{\mathcal{F}_t, t \in [0,T]\}$ denotes the natural filtration generated by the fractional noise B^H, that is, $\mathcal{F}_t$ is the completion of $\sigma\left\{B^H(s,x), s \le t, x \in D\right\}$, which thus satisfies the usual conditions.

Remark 2.5. *The following embedding lemma (see [14]) yields directly Proposition 2.3.*

Lemma 2.6. *If $h \in (\frac{1}{2}, 1)$ and $f, g \in L^{\frac{1}{h}}([a,b])$, then*

$$\int_a^b \int_a^b f(u)g(v)|u-v|^{2h-2}dudv \le C(h)\|f\|_{L^{\frac{1}{h}}([a,b])}\|g\|_{L^{\frac{1}{h}}([a,b])},$$

where $C(h) > 0$ is a constant depending only on h.

2.2 Several technical estimates

We are concerned with the following SPDE driven by a space-time fractional Brownian field:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + g(u(t,x), \frac{\partial}{\partial x}u(t,x)) + \dot{B}^H, \\ u(t,0) = 0, \\ u(0,x) = u_0(x), \end{cases} \tag{2.3}$$

for $(t,x) \in [0,T] \times D$, where

$$\dot{B}^H = B^H(dt,dx) = \frac{\partial^2 B^H(t,x)}{\partial t \partial x}$$

is a fractional Brownian field on $(\Omega, \mathcal{F}, \mathbb{P})$ with Hurst parameter $H = (h_1, h_2)$ for $h_i \in (0,1)$ and $i \in \{1,2\}$.

Throughout the remaining part of the paper, the Hurst parameter $H = (h_1, h_2)$ satisfies the following hypothesis H_{h_1,h_2}:

Hypothesis 2.1. *(1) $h_i \in (\frac{1}{2}, 1)$, $i = 1, 2$, and (2) $2h_1 + h_2 > 2$.*

The initial data $u_0 : D \mapsto \mathbb{R}$ satisfies the following hypothesis H_{u_0}:

Hypothesis 2.2. *(1) $||u_0||_\infty := \sup_x |u_0(x)| < \infty$,*
(2) $||u_0'||_\infty < \infty$,
(3) $u_0'(x)$ is κ-Hölder continuous in x with $\kappa \in (0, 1)$.

The function $g : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ satisfies the following hypothesis H_g:

Hypothesis 2.3. *There exists a constant $L > 0$ such that $|g(x_1, y_1) - g(x_2, y_2)| \leq L(|x_1 - x_2| + |y_1 - y_2|)$.*

Let us consider the following non-local SPDE:

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = \frac{1}{2}\Delta u(t, x) + g(u(t, x), u^\theta(t, x)) + \dot{B}^H, \\ u^\theta(t, x) = \frac{1}{\theta}\left(u(t, x + \theta) - u(t, x)\right), \\ u(t, 0) = 0, \\ u(0, x) = u_0(x) \end{cases} \tag{2.4}$$

where $\theta \in \mathbb{R}$, $\theta \neq 0$, $(t, x) \in [0, T] \times D$ and $(t, x + \theta) \in [0, T] \times D$.

Suppose $p(t, x, y)$ is the Green function of $\frac{\partial}{\partial t} - \frac{1}{2}\Delta$ in D subject to the Dirichlet boundary condition, that is,

$$p(t, x, y) = \frac{1}{\sqrt{2\pi t}}\left(e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}}\right),$$

then we may rewrite Eq. (2.4) as the following

$$\begin{cases} u(t, x) = \int_D p(t, x, y)u_0(y)dy + \int_0^t \int_D p(t - s, x, y)g(u(s, y), u^\theta(s, y))dyds \\ \qquad + \int_0^t \int_D p(t - s, x, y)B^H(ds, dy), \\ u^\theta(t, x) = \frac{1}{\theta}\left(u(t, x + \theta) - u(t, x)\right). \end{cases}$$

Note that, from (2.2),

$$\int_0^t \int_D p(t - s, x, y)B^H(ds, dy) = \int_0^t \int_D (K_H^* p)(t - s, x, y)B(ds, dy).$$

The following lemma provides the key estimate we need in what follows.

Lemma 2.7. *Suppose $\psi(t, x)$ is a measurable function, and suppose*

$$|H(t, x, y)| \leq t^{-\rho} e^{-\frac{C(x-y)^2}{t}},$$

where $\rho < \frac{3}{2}$. Then there exists a constant $C_T > 0$ such that for $t \in [0, T]$,

$$\mathbb{E}\left(\int_0^t \int_D H(t - s, x, y)\psi(s, y)dyds\right)^2 \leq C_T \int_0^t (t - s)^{\frac{1}{2} - \rho} \sup_y \mathbb{E}\left(\psi^2(s, y)\right) ds. \tag{2.5}$$

Proof. Applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
& \mathbb{E}\left(\int_0^t \int_D H(t-s,x,y)\psi(s,y)dyds\right)^2 \\
\leq & \left(\int_0^t \int_D H(t-s,x,y)dyds\right) \times \mathbb{E}\left(\int_0^t \int_D H(t-s,x,y)\psi^2(s,y)dyds\right) \\
\leq & C_T \int_0^t (t-s)^{\frac{1}{2}-\rho} \sup_y \mathbb{E}\left(\psi^2(s,y)\right) ds,
\end{aligned}$$

which completes the proof. ◻

Remark 2.8. *The previous estimate is applicable to*

$$H(t,x,y) = \frac{1}{\theta}\left(p(t,x+\theta,y) - p(t,x,y)\right),$$

so that

$$\mathbb{E}\left(\int_0^t \int_D H(t-s,x,y)\psi(s,y)dyds\right)^2 \leq C_T \int_0^t (t-s)^{-\frac{1}{2}} \sup_y \mathbb{E}\left(\psi^2(s,y)\right) ds.$$

In fact,

$$\begin{aligned}
H(t,x,y) &= \frac{1}{\theta}\left(p(t,x+\theta,y) - p(t,x,y)\right) \\
&= \int_0^1 \frac{\partial}{\partial x} p(t,x+a\theta,y)da,
\end{aligned}$$

and (see also Lemma 5.1)

$$\frac{\partial}{\partial x} p(t,x,y) \leq C t^{-1} e^{-\frac{(x-y)^2}{4t}}.$$

Thus, by the Fubini Theorem, we deduce that

$$\begin{aligned}
\int_0^t \int_D H(t-s,x,y)dyds &= \int_0^t \int_D \int_0^1 \frac{\partial}{\partial x} p(t,x+a\theta,y)dadyds \\
&= \int_0^1 \left(\int_0^t \int_D \frac{\partial}{\partial x} p(t,x+a\theta,y)dyds\right) da \\
&\leq C \int_0^1 t^{\frac{1}{2}} da \leq C_T.
\end{aligned}$$

Therefore

$$\begin{aligned}
& \mathbb{E}\left(\int_0^t \int_D H(t-s,x,y)\psi^2(s,y)dyds\right) \\
= & \mathbb{E}\left(\int_0^t \int_D \int_0^1 \frac{\partial}{\partial x} p(t,x+a\theta,y)\psi^2(s,y)dadyds\right) \\
= & \mathbb{E}\left(\int_0^1 \left(\int_0^t \int_D \frac{\partial}{\partial x} p(t,x+a\theta,y)\psi^2(s,y)dyds\right) da\right)
\end{aligned}$$

$$\leq \quad C_T \int_0^t (t-s)^{-\frac{1}{2}} \sup_y \mathbb{E}\left(\psi^2(s,y)\right) ds.$$

We also need an estimate on the second moment of some stochastic integrals.

Lemma 2.9. *Suppose $f(t,x) \in L_H^2$, then*

$$\mathbb{E}\left[\int_0^t \int_D f(s,x) B^H(dx,ds)\right]^2 \leq C(h_1,h_2)\left(\int_0^t (\|f(s,\cdot)\|_{L^{\frac{1}{h_2}}(D)})^{\frac{1}{h_1}} ds\right)^{2h_1}. \tag{2.6}$$

Proof. By Proposition 2.6, we have

$$\begin{aligned}
& \mathbb{E}\left[\int_0^t\int_D f(s,x) B^H(dx,ds)\right]^2 \\
= & \int_{[0,t]^2}\int_{D^2} \Psi_h(s_1,s_2,y_1,y_2) f(s_1,y_1) f(s_2,y_2) dy_1 dy_2 ds_1 ds_2 \\
= & \ C(h_1,h_2) \int_{[0,t]^2}\int_{D^2} |s_1-s_2|^{2h_1-2} |y_1-y_2|^{2h_2-2} f(s_1,y_1) f(s_2,y_2) dy_1 dy_2 ds_1 ds_2 \\
\leq & \ C(h_1,h_2) \int_{[0,t]^2} |s_1-s_2|^{2h_1-2} \|f(s_1,\cdot)\|_{L^{\frac{1}{h_2}}(D)} \|f(s_2,\cdot)\|_{L^{\frac{1}{h_2}}(D)} ds_1 ds_2 \\
\leq & \ C(h_1,h_2)\left(\int_0^t (\|f(s,\cdot)\|_{L^{\frac{1}{h_2}}(D)})^{\frac{1}{h_1}} ds\right)^{2h_1},
\end{aligned}$$

and the proof of this lemma is complete. □

3 Solvability of non-local SPDE

This section is devoted to the study of non-local SPDE. We study the uniqueness, existence and regularity of the solution.

3.1 Uniqueness

$\mathbb{S}$ denotes the collection of all functions $u : D \times [0,T] \to \mathbb{R}$ such that for every $t \in [0,T]$,

$$\sup_x \mathbb{E}|u(t,x)|^2 < \infty.$$

In fact, for the solution set $\mathbb{S}$: $\sup_x \mathbb{E}|u(t,x)|^2$ is locally integrable in t in order to apply Henry's Gronwall type inequality later.

Theorem 3.1. *Suppose H_{h_1,h_2}, H_{u_0} and H_g hold, then there exists at most one solution $(u(t,x), u^\theta(t,x))$ of the SPDE (2.4), where $u(t,x) \in \mathbb{S}$ and $u^\theta(t,x) \in \mathbb{S}$.*

Proof. Suppose $(u(t,x), u^\theta(t,x))$ and $(\tilde{u}(t,x), \tilde{u}^\theta(t,x))$ are two solutions of the equation (2.2), then

$$\begin{aligned}
& |u(t,x) - \tilde{u}(t,x)| \\
= & \left| \int_0^t \int_D p(t-s,x,y) \left[g(u(s,y), u^\theta(s,y)) - g(\tilde{u}(s,y), \tilde{u}^\theta(s,y)) \right] dyds \right| \\
\leq & C \int_0^t \int_D |p(t-s,x,y)| \left[|u(s,y) - \tilde{u}(s,y)| + |u^\theta(s,y) - \tilde{u}^\theta(s,y)| \right] dyds,
\end{aligned}$$

so that, by Lemma 2.7,

$$\begin{aligned}
\sup_x \mathbb{E}|u(t,x) - \tilde{u}(t,x)|^2 \leq & C \int_0^t \sup_y \mathbb{E}|u(s,y) - \tilde{u}(s,y)|^2 ds \\
& + C \int_0^t \sup_y \mathbb{E}|u^\theta(s,y) - \tilde{u}^\theta(s,y)|^2 ds. \qquad (3.1)
\end{aligned}$$

Since

$$\begin{aligned}
& u^\theta(t,x) - \tilde{u}^\theta(t,x) \\
= & \int_0^t \int_D \frac{p(t-s,x+\theta,y) - p(t-s,x,y)}{\theta} \\
& \times \left[g(u(s,y), u^\theta(s,y)) - g(\tilde{u}(s,y), \tilde{u}^\theta(s,y)) \right] dyds.
\end{aligned}$$

According to (2.5), we have

$$\begin{aligned}
& \sup_x \mathbb{E}|u^\theta(t,x) - \tilde{u}^\theta(t,x)|^2 \\
\leq & C \int_0^t (t-s)^{-\frac{1}{2}} \sup_y \mathbb{E}|u(s,y) - \tilde{u}(s,y)|^2 ds \\
& + C \int_0^t (t-s)^{-\frac{1}{2}} \sup_y \mathbb{E}|u^\theta(s,y) - \tilde{u}^\theta(s,y)|^2 ds. \qquad (3.2)
\end{aligned}$$

Let

$$\Gamma(t) = \sup_x \mathbb{E}|u(t,x) - \tilde{u}(t,x)|^2 + \sup_x \mathbb{E}|u^\theta(t,x) - \tilde{u}^\theta(t,x)|^2.$$

Then from (3.1) and (3.2), we have

$$\begin{aligned}
\Gamma(t) \leq & C \int_0^t \left(1 + (t-s)^{-\frac{1}{2}}\right) \sup_y \mathbb{E}|u(s,y) - \tilde{u}(s,y)|^2 ds \\
& + C \int_0^t \left(1 + (t-s)^{-\frac{1}{2}}\right) \sup_y \mathbb{E}|u^\theta(s,y) - \tilde{u}^\theta(s,y)|^2 ds \\
= & C \int_0^t \left(1 + (t-s)^{-\frac{1}{2}}\right) \Gamma(s) ds,
\end{aligned}$$

thus it follows from the Gronwall inequality (see e.g. Lemma 1.1 in [9]) that

$$\Gamma(t) = 0, \quad as \quad t \in [0,T].$$

Hence

$$\sup_x \mathbb{E}|u(t,x)-\tilde{u}(t,x)|^2 = 0, \quad as \quad t\in[0,T],$$

and therefore

$$\sup_x \mathbb{E}|u^\theta(t,x)-\tilde{u}^\theta(t,x)|^2 = 0, \quad as \quad t\in[0,T].$$

That is,

$$(u(t,x),u^\theta(t,x)) = (\tilde{u}(t,x),\tilde{u}^\theta(t,x)), \quad (t,x)\in[0,T]\times D,$$

in L^2 sense. The proof thus is completed. □

3.2 Existence

Theorem 3.2. *Suppose that H_{h_1,h_2}, H_{u_0} and H_g hold, then there exists one solution (u,u^θ) of the SPDE (2.4), where $u\in\mathbb{S}$ and $u^\theta\in\mathbb{S}$.*

To prove Theorem 3.2, let us consider the Picard iteration $\{(u_n(t,x),u_n^\theta(t,x))\}_{n\geq 0}$ defined by

$$\begin{cases} u_{n+1}(t,x) = \int_D p(t,x,y)u_0(y)dy + \int_0^t\int_D p(t-s,x,y)g(u_n(s,y),u_n^\theta(s,y))dsdy \\ \qquad\qquad + \int_0^t\int_D p(t-s,x,y)B^H(ds,dy), \\ u_n^\theta(t,x) = \frac{1}{\theta}\left(u_n(t,x+\theta)-u_n(t,x)\right) \end{cases}$$

where

$$u_0(t,x) := \int_D p(t,x,y)u_0(y)dy.$$

If $u_n(t,x)\in\mathbb{S}$, then clearly $u_n^\theta(t,x)\in\mathbb{S}$. We check that $u_{n+1}\in\mathbb{S}$. Note that

$$\mathbb{E}\left(\int_D p(t,x,y)u_0(y)dy\right)^2 \leq ||u_0||_\infty^2,$$

so that, by Lemma 2.7, we have

$$\begin{aligned} &\mathbb{E}\left(\int_0^t\int_D p(t-s,x,y)g(u_n(s,y),u_n^\theta(s,y))dsdy\right)^2 \\ \leq\ & C\left(1+\int_0^t\sup_y \mathbb{E}|u_n(s,y)|^2ds + \int_0^t\sup_y\mathbb{E}|u_n^\theta(s,y)|^2ds\right). \end{aligned}$$

While, by Lemma 2.9, one gets

$$\mathbb{E}\left(\int_0^t\int_D p(t-s,x,y)B^H(ds,dy)\right)^2 \leq C(h_1,h_2)\left(\int_0^t (\|p(t-s,x,\cdot)\|_{L^{\frac{1}{h_2}}(D)})^{\frac{1}{h_1}}ds\right)^{2h_1}$$

and

$$\begin{aligned} \|p(t-s,x,\cdot)\|_{L^{\frac{1}{h_2}}(D)} &= \left(\int_D |p(t-s,x,y)|^{\frac{1}{h_2}}dy\right)^{h_2} \\ &\leq C\left(\int_D (t-s)^{-\frac{1}{2h_2}}\exp(-\frac{1}{4h_2}\frac{|x-y|^2}{t-s})dy\right)^{h_2} \end{aligned}$$

$$
\begin{aligned}
&\leq C\left((t-s)^{\frac{1}{2}(1-\frac{1}{h_2})}\right)^{h_2} \\
&= C(t-s)^{\frac{1}{2}(h_2-1)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathbb{E}\left(\int_0^t \int_D p(t-s,x,y)B^H(ds,dy)\right)^2 &\leq C(h_1,h_2)\left(\int_0^t (\|p(t-s,x,\cdot)\|_{L^{\frac{1}{h_2}}(D)})^{\frac{1}{h_1}} ds\right)^{2h_1} \\
&\leq C(h_1,h_2)t^{2h_1+h_2-1} \\
&\leq C(h_1,h_2,T) \qquad (3.3)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\sup_x \mathbb{E}|u_{n+1}(t,x)|^2 &\leq C + C\int_0^t \sup_y \mathbb{E}|u_n(s,y)|^2 ds \\
&\quad + C\int_0^t \sup_y \mathbb{E}|u_n^\theta(s,y)|^2 ds. \qquad (3.4)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
u_{n+1}^\theta(t,x) &= \int_D \frac{1}{\theta}\left(p(t,x+\theta,y) - p(t,x,y)u_0(y)\right) dy \\
&\quad + \int_0^t \int_D \frac{1}{\theta}\left(p(t-s,x+\theta,y) - p(t-s,x,y)\right) g(u_n(s,y), u_n^\theta(s,y)) dyds \\
&\quad + \int_0^t \int_D \frac{1}{\theta}\left(p(t-s,x+\theta,y) - p(t-s,x,y)\right) B^H(ds,dy) \\
&= \int_D \int_0^1 \frac{\partial}{\partial x} p(t,x+a\theta,y)u_0(y) dady \\
&\quad + \int_0^t \int_D \int_0^1 \frac{\partial}{\partial x} p(t-s,x+a\theta,y) g(u_n(s,y), u_n^\theta(s,y)) dadyds \\
&\quad + \int_0^t \int_D \int_0^1 \frac{\partial}{\partial x} p(t-s,x+a\theta,y) da B^H(ds,dy),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E}\left(\int_D \int_0^1 \frac{\partial}{\partial x} p(t,x+a\theta,y)u_0(y) dady\right)^2 &= \mathbb{E}\left(\int_0^1 \left(-\int_D p(t,x+a\theta,y)u_0'(y)dy\right) da\right)^2 \\
&\leq ||u_0'||_\infty.
\end{aligned}
$$

By Hypothesis 2.3 and Remark 2.8, we thus obtain

$$
\begin{aligned}
&\mathbb{E}\left(\int_0^t \int_D \int_0^1 \frac{\partial}{\partial x} p(t-s,x+a\theta,y) g(u_n(s,y), u_n^\theta(s,y)) dadyds\right)^2 \\
&\leq C + C\int_0^t (t-s)^{-\frac{1}{2}} \sup_y \mathbb{E}|u_n(s,y)|^2 ds + C\int_0^t (t-s)^{-\frac{1}{2}} \sup_y \mathbb{E}|u_n^\theta(s,y)|^2 ds.
\end{aligned}
$$

By a similar argument as those in the proof of (3.3), we have

$$\begin{aligned}
&\mathbb{E}\left(\int_0^t\int_D\int_0^1 \frac{\partial}{\partial x}p(t-s,x+a\theta,y)daB^H(ds,dy)\right)^2\\
\leq\;& C(h_1,h_2)\left(\int_0^t\left(\left\|\int_0^1 \frac{\partial}{\partial x}p(t-s,x+a\theta,\cdot)da\right\|_{L^{\frac{1}{h_2}}(D)}\right)^{\frac{1}{h_1}}ds\right)^{2h_1}\\
\leq\;& C(h_1,h_2)\int_0^t (t-s)^{\frac{1}{2h_1}(h_2-2)}ds\\
\leq\;& C(h_1,h_2,T), \qquad (3.5)
\end{aligned}$$

where we have used the assumption that $2h_1+h_2-2>0$, the Fubini theorem and Lemma 5.1 which yields that

$$\begin{aligned}
&\left\|\int_0^1 \frac{\partial}{\partial x}p(t-s,x+a\theta,\cdot)da\right\|_{L^{\frac{1}{h_2}}(D)}\\
=\;& \left(\int_D\left|\int_0^1 \frac{\partial}{\partial x}p(t-s,x+a\theta,y)da\right|^{\frac{1}{h_2}}dy\right)^{h_2}\\
\leq\;& C\left(\int_D\int_0^1 \left|\frac{\partial}{\partial x}p(t-s,x+a\theta,y)\right|^{\frac{1}{h_2}}dady\right)^{h_2}\\
\leq\;& C\left(\int_0^1\left(\int_D (t-s)^{-\frac{1}{h_2}}\exp(-\frac{1}{4h_2}\frac{|x+a\theta-y|^2}{t-s})dy\right)da\right)^{h_2}\\
\leq\;& C\left((t-s)^{\frac{1}{2h_2}(h_2-2)}\right)^{h_2}\\
=\;& C(t-s)^{\frac{1}{2}(h_2-2)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\sup_x \mathbb{E}|u^\theta_{n+1}(t,x)|^2\\
\leq\;& C+C\left(\int_0^t (t-s)^{-\frac{1}{2}}\sup_y \mathbb{E}|u_n(s,y)|^2ds+\int_0^t (t-s)^{-\frac{1}{2}}\sup_y \mathbb{E}|u^\theta_n(s,y)|^2ds\right). \qquad (3.6)
\end{aligned}$$

Let

$$\Psi_n(t)=\sup_x \mathbb{E}|u_n(t,x)|^2+\sup_x \mathbb{E}|u^\theta_n(t,x)|^2$$

and

$$\Psi(t)=\limsup_n \Psi_n(t).$$

Then, by (3.4) and (3.6), we get

$$\Psi_{n+1}(t) \leq C+C\int_0^t\left(1+(t-s)^{-\frac{1}{2}}\right)\sup_y \mathbb{E}|u_n(s,y)|^2ds$$

$$
\begin{aligned}
&+C\int_0^t\left(1+(t-s)^{-\frac{1}{2}}\right)\sup_y \mathbb{E}|u_n^\theta(s,y)|^2ds \\
=&\quad C+C\int_0^t\left(1+(t-s)^{-\frac{1}{2}}\right)\Psi_n(s)ds,
\end{aligned}
$$

and therefore

$$
\Psi(t)\le C+C\int_0^t\left(1+(t-s)^{-\frac{1}{2}}\right)\Psi(s)ds.
$$

By applying the Gronwall inequality, to obtain that $\Psi(t)<\infty$ for $t\in[0,T]$. It follows that

$$
\sup_x \mathbb{E}|u_n(t,x)|^2<\infty,\quad as\quad t\in[0,T],
$$

and

$$
\sup_x \mathbb{E}|u_n^\theta(t,x)|^2<\infty,\quad as\quad t\in[0,T].
$$

Therefore, for any n and $(t,x)\in[0,T]\times D$,

$$
u_n(t,x)\in\mathbb{S},\quad u_n^\theta(t,x)\in\mathbb{S}.
$$

We next prove that the sequences $\{(u_n(t,x)\}_{n\ge0}$ and $\{(u_n^\theta(t,x)\}_{n\ge0}$ are Cauchy sequences in $\mathbb{S}$. To this end, consider

$$
\begin{cases}
u_{n+k+1}(t,x)-u_{n+1}(t,x)\\
\quad=\int_0^t\int_D p(t-s,x,y)\left[g(u_{n+k}(s,y),u_{n+k}^\theta(s,y))-g(u_n(s,y),u_n^\theta(s,y))\right]dyds\\
u_n^\theta(t,x)=\frac{1}{\theta}\left(u_n(t,x+\theta)-u_n(t,x)\right),
\end{cases}
$$

where $k=1,2,3,\cdots$. Then, there is a similar estimate as (3.4) for the difference

$$
\begin{aligned}
&\sup_x \mathbb{E}|u_{n+k+1}(t,x)-u_{n+1}(t,x)|^2\\
\le&\quad C\int_0^t\sup_y \mathbb{E}|u_{n+k}(s,y)-u_n(s,y)|^2ds\\
&+C\int_0^t\sup_y \mathbb{E}|u_{n+k}^\theta(s,y)-u_n^\theta(s,y)|^2ds.
\end{aligned}\tag{3.7}
$$

On the other hand

$$
\begin{aligned}
&\sup_x \mathbb{E}|u_{n+k+1}^\theta(t,x)-u_{n+1}^\theta(t,x)|^2\\
\le&\quad C\int_0^t(t-s)^{-\frac{1}{2}}\sup_y \mathbb{E}|u_{n+k}(s,y)-u_n(s,y)|^2ds\\
&+C\int_0^t(t-s)^{-\frac{1}{2}}\sup_y \mathbb{E}|u_{n+k}^\theta(s,y)-u_n^\theta(s,y)|^2ds.
\end{aligned}\tag{3.8}
$$

So that, by (3.7) and (3.8),

$$
\Phi(t)\le C\int_0^t\left(1+(t-s)^{-\frac{1}{2}}\right)\Phi(s)ds
$$

where

$$\Phi(t) = \limsup_{n\to\infty} \sup_k \left(\sup_x \mathbb{E}|u_{n+k+1}(t,x) - u_{n+1}(t,x)|^2 + \sup_x \mathbb{E}|u^\theta_{n+k+1}(t,x) - u^\theta_{n+1}(t,x)|^2 \right).$$

Once again by Gronwall inequality,

$$\Phi(t) = 0, \quad t \in [0,T].$$

Then

$$\sup_x \mathbb{E}|u_{n+k}(t,x) - u_n(t,x)|^2 \to 0$$

as $n \to \infty$, for every k and for all $t \in [0,T]$. Therefore

$$\sup_x \mathbb{E}|u^\theta_{n+k}(t,x) - u^\theta_n(t,x)|^2 \to 0$$

as $n \to \infty$ for $t \in [0,T]$. That is, $\{u_n(t,x)\}$ and $\{u^\theta_n(t,x)\}$ are Cauchy sequences on $\mathbb{S}$. The limits of these sequences are denoted by $u(t,x)$ and $u^\theta(t,x)$ which also belong to $\mathbb{S}$. Therefore the pair $(u(t,x), u^\theta(t,x))$ is a solution of the SPDE (2.4).

3.3 Regularity of the unique solution

Let $(u(t,x), u^\theta(t,x))$ be the solution of the stochastic equation (2.4) under the assumptions as in Theorem 3.2. Then $u(t,x) \in \mathbb{S}$ and $u^\theta(t,x) \in \mathbb{S}$. We next discuss the Hölder continuity of $u(t,x)$ and $u^\theta(t,x)$.

Theorem 3.3. *Assume that H_{h_1,h_2}, H_{u_0} and H_g hold, and that $u(t,x)$ is the solution of the equation (2.4). Then $u(t,x)$ is μ_1-Hölder continuous in t and ν_1-Hölder continuous in x, where $\mu_1 \in (0, \frac{1}{2})$ and $\nu_1 \in (0,1)$. Moreover, $u^\theta(t,x)$ is μ_2-Hölder continuous in t and ν_2-Hölder continuous in x, where $\mu_2 \in (0, \min\{\frac{\kappa}{2}, \frac{2h_1+h_2-1}{3}\})$ and $\nu_2 \in (0, \min\{\kappa, \frac{2h_1+h_2-1}{2}\})$.*

The remainder of the section is devoted to the proof of the theorem above.

Without loss of generality, we suppose that $0 \le s \le t \le T$ and $0 \le y \le x$. First observe that

$$\mathbb{E}(u(t,x) - u(s,y))^2 \le 2\left(\mathbb{E}(u(t,x) - u(t,y))^2 + \mathbb{E}(u(t,y) - u(s,y))^2\right).$$

It is elementary to see that

$$\begin{aligned}
& \mathbb{E}(u(t,x) - u(t,y))^2 \\
\le\ & C\left(\mathbb{E}\left[\int_D (p(t,x,z) - p(t,y,z))u_0(z)dz\right]\right)^2 \\
& + \mathbb{E}\left(\int_0^t \int_D (p(t-r,x,z) - p(t-r,y,z))g(u(r,z), u^\theta(r,z))dzdr\right)^2 \\
& + \mathbb{E}\left(\int_0^t \int_D (p(t-r,x,z) - p(t-r,y,z))B^H(dr,dz)\right)^2 \\
=\ & C(I_1 + I_2 + I_3).
\end{aligned}$$

According to Hypothesis 2.2, we have

$$\begin{aligned} |u_0(x) - u_0(y)| &= |u_0^{'}(\cdot)||x-y| \\ &\leq \|u_0^{'}\|_\infty |x-y|. \end{aligned}$$

By Lemma 5.2,

$$I_1 = \mathbb{E}\left(\int_D (p(t,x,z) - p(t,y,z))u_0(z)dz\right)^2 \leq C|x-y|^2.$$

Let us deal with I_2. Clearly

$$\begin{aligned} & \left|\int_0^t \int_D |p(t-r,x,z) - p(t-r,y,z)|\, g(u(r,z), u^\theta(r,z))dyds\right| \\ \leq\ & C\int_0^t \int_D |p(t-r,x,z) - p(t-r,y,z)| \left(1 + |u(r,z)| + |u^\theta(r,z)|\right)|dzdr \\ \leq\ & C\left\{\int_0^t \int_D |p(t-r,x,z) - p(t-r,y,z)|\, dzdr \right. \\ & + \int_0^t \int_D |p(t-r,x,z) - p(t-r,y,z)|\, |u(r,z)|dzdr \\ & \left. + \int_0^t \int_D |p(t-r,x,z) - p(t-r,y,z)|\, |u^\theta(r,z)|dzdr \right\}. \end{aligned}$$

While, by Lemma 5.1 and the Fubini theorem, one gets

$$\begin{aligned} & \mathbb{E}\left(\int_0^t \int_D |p(t-r,x,z) - p(t-r,y,z)|\, |u(r,z)|dzdr\right)^2 \\ \leq\ & \int_0^t \int_D |p(t-r,x,z) - p(t-r,y,z)|\, dzdr \\ & \times \int_0^t \int_D |p(t-r,x,z) - p(t-r,y,z)|\, |u(r,z)|^2 dzdr \\ \leq\ & C\left(\int_0^t \int_D |p(t-r,x,z) - p(t-r,y,z)|\right)^2 dzdr \\ =\ & C|x-y|^2 \left(\int_0^t \int_D \left|\int_0^1 \frac{\partial}{\partial x} p(t-r, y + a(x-y), z)da\right| dzdr\right)^2 \\ \leq\ & C|x-y|^2 \left(\int_0^1 \left(\int_0^t \int_D (t-r)^{-1} e^{-\frac{(y+a(x-y)-z)^2}{4(t-r)}} dzdr\right) da\right)^2 \\ \leq\ & C|x-y|^2, \end{aligned}$$

and

$$\begin{aligned} & \int_0^t \int_D |p(t-r,x,z) - p(t-r,y,z)|\, |u^\theta(r,z)|dzdr \\ =\ & (x-y)\int_0^t \int_D \left|\int_0^1 \frac{\partial}{\partial x} p(t-r, y + a(x-y), z)da\right| |u^\theta(r,z)|dzdr. \end{aligned}$$

Thus, by using Lemma 5.1 and Remark 2.8, we have

$$\begin{aligned}
&\mathbb{E}\left(\int_0^t\int_D |p(t-r,x,z)-p(t-r,y,z)||u^\theta(r,z)|dzdr\right)^2\\
\leq\ & C|x-y|^2\int_0^t (t-r)^{-\frac{1}{2}}\sup_z \mathbb{E}|u^\theta(r,z)|^2dr\\
\leq\ & C|x-y|^2,
\end{aligned}$$

and therefore,

$$I_2\leq C|x-y|^2.$$

Next we estimate I_3. Let $\gamma\in(0,\min\{2h_1+h_2-1,1\})=(0,1)$. Then

$$\begin{aligned}
I_3 &= \mathbb{E}\left(\int_0^t\int_D (p(t-r,x,z)-p(t-r,y,z))B^H(dr,dz)\right)^2\\
&= \int_{[0,s]^2}\int_{D^2}\Psi_h(r,\bar r,z,\bar z)|p(t-r,x,z)-p(t-r,y,z)|\\
&\quad \times|p(t-\bar r,x,\bar z)-p(t-\bar r,y,\bar z)|dzd\bar z drd\bar r\\
&= \|p(t-\cdot,x,\cdot)-p(t-\cdot,y,\cdot)\|^2_{L^2_H}\\
&= \left\||p(t-\cdot,x,\cdot)-p(t-\cdot,y,\cdot)|^\gamma|p(t-\cdot,x,\cdot)-p(t-\cdot,y,\cdot)|^{1-\gamma}\right\|^2_{L^2_H}\\
&\leq C(\gamma)\Big(\left\||p(t-\cdot,x,\cdot)-p(t-\cdot,y,\cdot)|^\gamma|p(t-\cdot,x,\cdot)|^{1-\gamma}\right\|^2_{L^2_H}\\
&\quad +\left\||p(t-\cdot,x,\cdot)-p(t-\cdot,y,\cdot)|^\gamma|p(t-\cdot,y,\cdot)|^{1-\gamma}\right\|^2_{L^2_H}\Big)\\
&:= C(\gamma)(I_{31}+I_{32}).
\end{aligned}$$

On other hand, according to Lemma 5.1 and the Fubini theorem, one can get

$$\begin{aligned}
I_{31} &\leq \left\|\left|\int_0^1\frac{\partial}{\partial x}p(t-\cdot,y+a(x-y),\cdot)da\right|^\gamma|x-y|^\gamma|p(t-\cdot,x,\cdot)|^{1-\gamma}\right\|^2_{L^2_H}\\
&= |x-y|^{2\gamma}\int_{[0,T]^2}\int_{D^2}\left|\int_0^1\frac{\partial}{\partial x}p(t-r,y+a(x-y),z)da\right|^\gamma|p(t-r,x-z)|^{1-\gamma}\\
&\quad\times\Psi_h(r,\bar r,z,\bar z)\left|\int_0^1\frac{\partial}{\partial x}p(t-\bar r,y+a(x-y),\bar z)da\right|^\gamma|p(t-\bar r,x-\bar z)|^{1-\gamma}dzd\bar z drd\bar r\\
&\leq C(h_1,h_2,\gamma)|x-y|^{2\gamma}\\
&\quad\times\left(\int_0^T\left(\int_D\left(\left|\int_0^1\frac{\partial}{\partial x}p(t-r,y+a(x-y),z)da\right|^\gamma|p(t-r,x-z)|^{1-\gamma}\right)^{\frac{1}{h_2}}dz\right)^{\frac{h_2}{h_1}}dr\right)^{2h_1}\\
&\leq C(h_1,h_2,\gamma)|x-y|^{2\gamma}\int_0^1\left(\int_0^T(t-r)^{\frac{-\gamma-\frac{1}{2}(1-\gamma)}{h_1}}(t-r)^{\frac{h_2}{2h_1}}dr\right)^{2h_1}da\\
&\leq C(h_1,h_2,\gamma)|x-y|^{2\gamma}\left(\int_0^T(t-r)^{\frac{h_2-1-\gamma}{2h_1}}dr\right)^{2h_1}
\end{aligned}$$

$$\leq \quad C(h_1,h_2,\gamma)|x-y|^{2\gamma}.$$

Similarly, we have

$$I_{32} \leq C(T,h_1,h_2)|x-y|^{2\gamma}.$$

Therefore we deduce that

$$I_3 \leq C(T,h_1,h_2)|x-y|^{2\gamma}, \tag{3.9}$$

hence

$$\mathbb{E}(u(t,x)-u(t,y))^2 \leq C|x-y|^{2\nu_1}$$

for $\nu_1 \in (0,\min\{2h_1+h_2-1,1\}) = (0,1)$.

By a similar argument as above, we can get

$$\mathbb{E}(u(t,y)-u(s,y))^2 \leq C|t-s|^{2\mu_1}$$

for $\mu_1 \in (0,\frac{1}{2}\min\{2h_1+h_2-1,1\}) = (0,\frac{1}{2})$.

That is, $u(t,x)$ is μ_1-Hölder continuous in t and ν_1-Hölder continuous in x, where $\mu_1 \in (0,\frac{1}{2})$ and $\nu_1 \in (0,1)$.

On the other hand, for any $\theta > 0$, we recall

$$u^\theta(t,x) = \frac{1}{\theta}(u(t,x+\theta)-u(t,x)),$$

and $u^\theta(t,x) \in \mathbb{S}$, and

$$\begin{aligned}
u^\theta(t,x) &= \int_D \frac{1}{\theta}\left(p(t,x+\theta,z)-p(t,x,z)\right)u_0(z)dz \\
&\quad + \int_0^t\int_D \frac{1}{\theta}\left(p(t-s,x+\theta,z)-p(t-s,x,z)\right)g(u(s,z),u^\theta(s,z))dzds \\
&\quad + \int_0^t\int_D \frac{1}{\theta}\left(p(t-s,x+\theta,z)-p(t-s,x,z)\right)B^H(ds,dz) \\
&= \int_D\int_0^1 \frac{\partial}{\partial x}p(t,x+a\theta,z)u_0(z)dadz \\
&\quad + \int_0^t\int_D\int_0^1 \frac{\partial}{\partial x}p(t-s,x+a\theta,z)g(u(s,z),u^\theta(s,z))dadzds \\
&\quad + \int_0^t\int_D\int_0^1 \frac{\partial}{\partial x}p(t-s,x+a\theta,z)daB^H(ds,dz).
\end{aligned} \tag{3.10}$$

Thus

$$\begin{aligned}
& u^\theta(t,x)-u^\theta(t,y) \\
=\; & \int_D\left(\int_0^1 \frac{\partial}{\partial x}p(t,x+a\theta,z)da - \int_0^1 \frac{\partial}{\partial x}p(t,y+a\theta,z)da\right)u_0(z)dz \\
& + \int_0^t\int_D\left(\int_0^1 \frac{\partial}{\partial x}p(t-r,x+a\theta,z)da - \int_0^1 \frac{\partial}{\partial x}p(t-r,y+a\theta,z)da\right)g(u(r,z),u^\theta(r,z))dzdr
\end{aligned}$$

$$+\int_0^t\int_D\left(\int_0^1\frac{\partial}{\partial x}p(t-r,x+a\theta,z)da-\int_0^1\frac{\partial}{\partial x}p(t-r,y+a\theta,z)da\right)B^H(dr,dz).$$

By Hypothesis 2.2 and Lemma 5.2, one gets

$$\begin{aligned}
&\mathbb{E}\left(\int_D\left(\int_0^1\frac{\partial}{\partial x}p(t,x+a\theta,z)da-\int_0^1\frac{\partial}{\partial x}p(t,y+a\theta,z)da\right)u_0(z)dz\right)^2 \qquad (3.11)\\
=\ &\mathbb{E}\left(\int_0^1\left(-\int_D\left(p(t,x+a\theta,z)-p(t,y+a\theta,z)\right)u_0^{'}(z)dz\right)da\right)^2\\
\le\ &C|x-y|^{2\kappa}.
\end{aligned}$$

Moreover

$$\begin{aligned}
&\mathbb{E}\left(\int_0^t\int_D\left(\int_0^1\frac{\partial}{\partial x}p(t-r,x+a\theta,z)da-\int_0^1\frac{\partial}{\partial x}p(t-r,y+a\theta,z)da\right)g(u(r,z),u^\theta(r,z))dzdr\right)^2\\
\le\ &\int_0^t\int_D\left|\int_0^1\frac{\partial}{\partial x}p(t-r,x+a\theta,z)da-\int_0^1\frac{\partial}{\partial x}p(t-r,y+a\theta,z)da\right|dzdr\\
&\times\int_0^t\int_D\left|\int_0^1\frac{\partial}{\partial x}p(t-r,x+a\theta,z)da-\int_0^1\frac{\partial}{\partial x}p(t-r,y+a\theta,z)da\right|\mathbb{E}(g(u(r,z),u^\theta(r,z)))^2dzdr\\
\le\ &C\left(\int_0^t\int_D\left|\int_0^1\frac{\partial}{\partial x}p(t-r,x+a\theta,z)da-\int_0^1\frac{\partial}{\partial x}p(t-r,y+a\theta,z)da\right|dzdr\right)^2\\
=\ &C\left\{\int_0^t\int_D\left|\int_0^1\frac{\partial}{\partial x}p(t-r,x+a\theta,z)da-\int_0^1\frac{\partial}{\partial x}p(t-r,y+a\theta,z)da\right|^\varrho\right.\\
&\left.\times\left|\int_0^1\frac{\partial}{\partial x}p(t-r,x+a\theta,z)da-\int_0^1\frac{\partial}{\partial x}p(t-r,y+a\theta,z)da\right|^{1-\varrho}dzdr\right\}^2\\
\le\ &C|x-y|^{2\varrho}\left\{\int_0^t\int_D\left|\int_0^1\int_0^1\frac{\partial^2}{\partial x^2}p(t-r,y+b(x-y)+a\theta,z)dadb\right|^\varrho\right.\\
&\left.\times\left|\int_0^1\frac{\partial}{\partial x}p(t-r,x+a\theta,z)da-\int_0^1\frac{\partial}{\partial x}p(t-r,y+a\theta,z)da\right|^{1-\varrho}dzdr\right\}^2\\
\le\ &C|x-y|^{2\varrho}\left\{\left[\int_0^t\int_D\left|\int_0^1\int_0^1\frac{\partial^2}{\partial x^2}p(t-r,y+b(x-y)+a\theta,z)dadb\right|^\varrho\right.\right.\\
&\left.\times\left|\int_0^1\frac{\partial}{\partial x}p(t-r,x+a\theta,z)da\right|^{1-\varrho}dzdr\right]^2\\
&\left.+\left[\int_0^t\int_D\left|\int_0^1\int_0^1\frac{\partial^2}{\partial x^2}p(t-r,y+b(x-y)+a\theta,z)dadb\right|^\varrho\left|\int_0^1\frac{\partial}{\partial x}p(t-r,y+a\theta,z)da\right|^{1-\varrho}dzdr\right]^2\right\},
\end{aligned}$$

where $\varrho\in(0,1)$.

While, from Lemma 5.1 and Fubini theorem,

$$\int_0^t\int_D\left|\int_0^1\int_0^1\frac{\partial^2}{\partial x^2}p(t-r,y+b(x-y)+a\theta,z)dadb\right|^\varrho\left|\int_0^1\frac{\partial}{\partial x}p(t-r,x+a\theta,z)da\right|^{1-\varrho}dzdr$$

$$
\begin{aligned}
&\leq C\int_0^t\int_D\int_0^1\int_0^1\int_0^1 \left|\frac{\partial^2}{\partial x^2}p(t-r,y+b(x-y)+a\theta,z)\right|^{\varrho}\left|\frac{\partial}{\partial x}p(t-r,x+c\theta,z)\right|^{1-\varrho} dadbdcdzdr \\
&\leq C\int_0^1\int_0^1\int_0^1\left(\int_0^t\int_D\left|(t-r)^{-\frac{3}{2}}e^{-\frac{(y+b(x-y)+a\theta-z)^2}{4t}}\right|^{\varrho}\left|(t-r)^{-1}e^{-\frac{(x+c\theta-z)^2}{4(t-r)}}\right|^{1-\varrho}dzdr\right)dadbdc \\
&\leq C\int_0^1\int_0^1\int_0^1\left(\int_0^t\int_D(t-r)^{-1-\frac{1}{2}\varrho}e^{-\frac{(y+b(x-y)+a\theta-z)^2}{4(t-r)}}drdz\right)dadbdc \\
&\leq C\int_0^t(t-r)^{-\frac{1}{2}-\frac{1}{2}\varrho}dr \\
&< \infty,
\end{aligned}
$$

with $-\frac{1}{2}-\frac{1}{2}\varrho>-1$ as $\varrho<1$.

Similarly, by using the argument as in the proof of (3.9), we have

$$
\begin{aligned}
&\mathbb{E}\left(\int_0^t\int_D\left(\int_0^1\frac{\partial}{\partial x}p(t-r,x+a\theta,z)da-\int_0^1\frac{\partial}{\partial x}p(t-r,y+a\theta,z)da\right)B^H(dr,dz)\right)^2 \\
&= \left\|\int_0^1\frac{\partial}{\partial x}p(t-\cdot,x+a\theta,\cdot)da-\int_0^1\frac{\partial}{\partial x}p(t-\cdot,y+a\theta,\cdot)da\right\|^2_{L^2_H} \\
&= \Bigg\|\left|\int_0^1\frac{\partial}{\partial x}p(t-\cdot,x+a\theta,\cdot)da-\int_0^1\frac{\partial}{\partial x}p(t-\cdot,y+a\theta,\cdot)da\right|^{\gamma'} \\
&\quad\times\left|\int_0^1\frac{\partial}{\partial x}p(t-\cdot,x+a\theta,\cdot)da-\int_0^1\frac{\partial}{\partial x}p(t-\cdot,y+a\theta,\cdot)da\right|^{1-\gamma'}\Bigg\|^2_{L^2_H} \\
&\leq C(h_1,h_2,\gamma)|x-y|^{2\gamma'}
\end{aligned}
$$

where $\gamma'\in(0,\min\{\frac{2h_1+h_2-1}{2},1\})=(0,\frac{2h_1+h_2-1}{2})$.

Putting together the estimates above, we deduce that

$$
\mathbb{E}(u^\theta(t,x)-u^\theta(t,y))^2\leq C|x-y|^{2\nu_2} \tag{3.12}
$$

where $\nu_2\in(0,\min\{\kappa,\frac{2h_1+h_2-1}{2},1\})=(0,\min\{\kappa,\frac{2h_1+h_2-1}{2}\})$.

Next let us deal with the difference

$$
\begin{aligned}
&u^\theta(t,x)-u^\theta(s,x) \\
=& \int_D\left(\int_0^1\frac{\partial}{\partial x}p(t,x+a\theta,z)da-\int_0^1\frac{\partial}{\partial x}p(s,x+a\theta,z)da\right)u_0(z)dz \\
&+\int_s^t\int_D\int_0^1\frac{\partial}{\partial x}p(t-r,x+a\theta,z)g(u(r,z),u^\theta(r,z))dadzdr \\
&+\int_0^s\int_D\left(\int_0^1\frac{\partial}{\partial x}p(t-r,x+a\theta,z)da-\int_0^1\frac{\partial}{\partial x}p(s-r,x+a\theta,z)da\right)g(u(r,z),u^\theta(r,z))dzdr \\
&+\int_0^t\int_D\left(\int_0^1\frac{\partial}{\partial x}p(t-r,x+a\theta,z)da-\int_0^1\frac{\partial}{\partial x}p(s-r,x+a\theta,z)da\right)B^H(dr,dz).
\end{aligned}
$$

Using the same approach to (3.11), together with Lemma 5.2 and the Fubini theorem, we obtain

$$\mathbb{E}\left[\left(\int_0^1 \frac{\partial}{\partial x}p(t,x+a\theta,z)da - \int_0^1 \frac{\partial}{\partial x}p(s,x+a\theta,z)da\right)u_0(z)dz\right]^2 \le C|t-s|^{\kappa}.$$

On the other hand, by Lemma 5.1,

$$\left|\frac{\partial^2}{\partial x\partial t}p(t,x,y)\right| \le Ct^{-2}e^{-\frac{(x-y)^2}{4t}}.$$

So that

$$\mathbb{E}\left(\int_s^t\int_D\int_0^1 \frac{\partial}{\partial x}p(t-r,x+a\theta,z)g(u(r,z),u^{\theta}(r,z))dadzdr\right)^2 \le C|t-s|^{2\sigma},$$

$$\begin{aligned}&\mathbb{E}\left(\int_0^s\int_D\int_0^1 \left(\frac{\partial}{\partial x}p(t-r,x+a\theta,z) - \frac{\partial}{\partial x}p(s-r,x+a\theta,z)\right)g(u(r,z),u^{\theta}(r,z))dadzdr\right)^2\\ \le\;& C|t-s|^{2\sigma}\end{aligned}$$

and

$$\begin{aligned}&\mathbb{E}\left(\int_0^1 \left(\frac{\partial}{\partial x}p(t-r,x+a\theta,z) - \frac{\partial}{\partial x}p(s-r,x+a\theta,z)\right)daB^H(dr,dz)\right)^2\\ \le\;& C(T,h_1,h_2)|t-s|^{2\iota},\end{aligned}$$

where $\sigma \in (0,\frac{1}{2})$ and $\iota \in \frac{2h_1+h_2-1}{3}$. Therefore

$$\mathbb{E}(u^{\theta}(t,x)-u^{\theta}(s,x))^2 \le C|t-s|^{2\mu_2},$$

where

$$\mu_2 \in \left\{0, \min\left(\frac{\kappa}{2},\frac{1}{2},\frac{2h_1+h_2-1}{3}\right)\right\} = \left\{0, \min\left(\frac{\kappa}{2},\frac{2h_1+h_2-1}{3}\right)\right\}.$$

Thus we finish the proof of the theorem.

4 Well Solvability of SPDE

In this part, we study SPDE (2.3). Let $v(t,x) = \frac{\partial}{\partial x}u(t,x)$. Then the SPDE (2.3) has the following equivalence expression:

$$\left\{\begin{array}{l}\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + g(u(t,x),v(t,x)) + \dot{B}^H,\\ u(t,0)=0,\\ u(0,x)=u_0(x),\end{array}\right. \tag{4.1}$$

which in turn means that the pair $(u(t,x),v(t,x))$ satisfies the coupled stochastic integral system:

$$\left\{\begin{array}{l}u(t,x) = \int_D p(t,x,y)u_0(y)dy + \int_0^t\int_D p(t-s,x,y)g(u(s,y),v(s,y))dyds\\ \qquad + \int_0^t\int_D p(t-s,x,y)B^H(ds,dy),\\ v(t,x) = \int_D \frac{\partial}{\partial x}p(t,x,y)u_0(y)dy + \int_0^t\int_D \frac{\partial}{\partial x}p(t-s,x,y)g(u(s,y),v(s,y))dyds\\ \qquad + \int_0^t\int_D \frac{\partial}{\partial x}p(t-s,x,y)B^H(ds,dy).\end{array}\right. \tag{4.2}$$

4.1 Existence

Theorem 4.1. *Suppose the assumptions H_{h_1,h_2}, H_{u_0} and H_g hold, then SPDE (2.3) possesses a solution in $\mathbb{S}$.*

In fact, we only need to show that SPDE (4.1) possesses a solution $(\bar{u}(t,x), \bar{v}(t,x))$, where $\bar{u}(t,x) \in \mathbb{S}$ and $\bar{v}(t,x) \in \mathbb{S}$.

As we have demonstrated, if$(u(t,x), u^\theta(t,x))$ is the solution of the equation (2.4), then $u(t,x) \in \mathbb{S}$ and $u^\theta(t,x) \in \mathbb{S}$.

Let us consider

$$\mathbb{E}(u^{\theta_1}(t,x) - u^{\theta_2}(t,x))^2$$

as $\theta_1 \to 0$ and $\theta_2 \to 0$.

According to (3.6), for $\tau_1, \tau_2 \in (0,1)$, we may write

$$\begin{aligned}
& u^{\theta_1}(t,x) - u^{\theta_2}(t,x) \\
= \; & \int_D \left(\int_0^1 \frac{\partial}{\partial x} p(t, x + a\theta_1, y) da - \int_0^1 \frac{\partial}{\partial x} p(t, x + a\theta_2, y) da \right) u_0(y) dy \\
& + \int_0^t \int_D \left\{ \int_0^1 \frac{\partial}{\partial x} p(t-s, x + a\theta_1, y) g(u(s,y), u^{\theta_1}(s,y)) da \right. \\
& \qquad \left. - \int_0^1 \frac{\partial}{\partial x} p(t-s, x + a\theta_2, y) g(u(s,y), u^{\theta_2}(s,y)) da \right\} dyds \\
& + \int_0^t \int_D \left\{ \int_0^1 \frac{\partial}{\partial x} p(t-s, x + a\theta_1, y) da - \int_0^1 \frac{\partial}{\partial x} p(t-s, x + a\theta_2, y) da \right\} B^H(ds, dy) \\
:= \; & B_1 + B_2 + B_3.
\end{aligned}$$

Let us derive estimates for B_1, B_2 and B_3. Note that

$$\begin{aligned}
& \int_0^1 \frac{\partial}{\partial x} p(t-s, x + a\theta_1, y) g(u(s,y), u^{\theta_1}(s,y)) da \\
& - \int_0^1 \frac{\partial}{\partial x} p(t-s, x + a\theta_2, y) g(u(s,y), u^{\theta_2}(s,y)) da \\
= \; & \int_0^1 \frac{\partial}{\partial x} p(t-s, x + a\theta_1, y) da \left(g(u(s,y), u^{\theta_1}(s,y)) - g(u(s,y), u^{\theta_2}(s,y)) \right) \\
& + \left(\int_0^1 \frac{\partial}{\partial x} p(t-s, x + a\theta_1, y) da - \int_0^1 \frac{\partial}{\partial x} p(t-s, x + a\theta_2, y) da \right) g(u(s,y), u^{\theta_2}(s,y))
\end{aligned}$$

and it follows that

$$\begin{aligned}
B_2 \; = \; & \int_0^t \int_D \int_0^1 \frac{\partial}{\partial x} p(t-s, x + a\theta_1, y) \left(g(u(s,y), u^{\theta_1}(s,y)) - g(u(s,y), u^{\theta_2}(s,y)) \right) dadyds \\
& + \int_0^t \int_D \left(\int_0^1 \frac{\partial}{\partial x} p(t-s, x + a\theta_1, y) da - \int_0^1 \frac{\partial}{\partial x} p(t-s, x + a\theta_2, y) da \right) g(u(s,y), u^{\theta_2}(s,y)) dyds \\
:= \; & B_{21} + B_{22}.
\end{aligned}$$

By the same argument as in the proof of (3.12), we may conclude that

$$B_1 \leq C|\theta_1 - \theta_2|^{2\kappa},$$

$$B_{22} \le C|\theta_1 - \theta_2|^{2\varrho},$$

and

$$B_3 \le C|\theta_1 - \theta_2|^{2\gamma'}.$$

Again by Lemma 5.1, one gets

$$B_{21} \le C \int_0^t (t-s)^{-\frac{1}{2}} \sup_y \mathbb{E}\left(u^{\theta_1}(s,y) - u^{\theta_2}(s,y)\right)^2 ds.$$

Combining the estimates above, we have

$$\begin{aligned} & \mathbb{E}\left(u^{\theta_1}(t,x) - u^{\theta_2}(t,x)\right)^2 \\ \le\ & C\,|\theta_1 - \theta_2|^{\varpi} + C \int_0^t (t-s)^{-\frac{1}{2}} \sup_y \mathbb{E}\left(u^{\theta_1}(s,y) - u^{\theta_2}(s,y)\right)^2 ds, \end{aligned} \tag{4.3}$$

where $\varpi = \min\{2\kappa, 2\varrho, 2\gamma'\}$.

Sending $\theta_1 \to 0$ and $\theta_2 \to 0$ and using Gronwall inequality, we get

$$\sup_x \mathbb{E}\left(u^{\theta_1}(t,x) - u^{\theta_2}(t,x)\right)^2 \to 0, \quad t \in [0,T].$$

Therefore, $\{u^\theta(t,x)\}_\theta$ is a Cauchy sequence on $\mathbb{S}$. The limit of these sequences exists (also belong to $\mathbb{S}$), which is $\bar{v}(t,x)$.

Finally letting $\theta \to 0$, we denote the limit of $(u(t,x), u^\theta(t,x))$ by $(\bar{u}(t,x), \bar{v}(t,x))$, which is the solution of the SPDE (4.1). The proof of this theorem is thus complete.

4.2 Uniqueness

Theorem 4.2. *Suppose the assumptions H_{h_1,h_2}, H_{u_0} and H_g hold, then SPDE (2.3) has a unique solution in $\mathbb{S}$.*

Proof. Suppose $(\bar{u}_1(t,x), \bar{v}_1(t,x))$ and $(\bar{u}_2(t,x), \bar{v}_2(t,x))$ are two solutions on the equation (4.2). Then

$$\begin{aligned} \bar{u}_1(t,x) &= \int_D p(t,x,y)u_0(y)dy + \int_0^t \int_D p(t-s,x,y) g(\bar{u}_1(s,y), \bar{v}_1(s,y))dyds \\ &\quad + \int_0^t \int_D p(t-s,x,y) B^H(ds,dy), \end{aligned}$$

and

$$\begin{aligned} \bar{u}_2(t,x) &= \int_D p(t,x,y)u_0(y)dy + \int_0^t \int_D p(t-s,x,y) g(\bar{u}_2(s,y), \bar{v}_2(s,y))dyds \\ &\quad + \int_0^t \int_D p(t-s,x,y) B^H(ds,dy). \end{aligned}$$

Then

$$\bar{u}_1(t,x) - \bar{u}_2(t,x)$$

$$= \int_0^t \int_D p(t-s,x,y) \left(g(\bar{u}_1(s,y), \bar{v}_1(s,y)) - g(\bar{u}_2(s,y), \bar{v}_2(s,y)) \right) dyds.$$

Once again by using Lemma 5.1 and Lemma 2.7,

$$\begin{aligned} & \sup_x \mathbb{E}(\bar{u}_1(t,x) - \bar{u}_2(t,x))^2 \\ \leq & \int_0^t \sup_y \mathbb{E}(\bar{u}_1(s,y) - \bar{u}_2(s,y))^2 ds + \int_0^t \sup_y \mathbb{E}(\bar{v}_1(s,y) - \bar{v}_2(s,y))^2 ds. \end{aligned} \tag{4.4}$$

On the other hand,

$$\begin{aligned} \bar{v}_1(t,x) &= \int_D \frac{\partial}{\partial x} p(t,x,y) u_0(y) dy + \int_0^t \int_D \frac{\partial}{\partial x} p(t-s,x,y) g(\bar{u}_1(s,y), \bar{v}_1(s,y)) dyds \\ &\quad + \int_0^t \int_D \frac{\partial}{\partial x} p(t-s,x,y) B^H(ds,dy), \end{aligned}$$

and

$$\begin{aligned} \bar{v}_2(t,x) &= \int_D \frac{\partial}{\partial x} p(t,x,y) u_0(y) dy + \int_0^t \int_D \frac{\partial}{\partial x} p(t-s,x,y) g(\bar{u}_2(s,y), \bar{v}_2(s,y)) dyds \\ &\quad + \int_0^t \int_D \frac{\partial}{\partial x} p(t-s,x,y) B^H(ds,dy). \end{aligned}$$

Note that

$$\bar{v}_i(t,x) = \lim_{\theta \to 0} \bar{u}_i^\theta(t,x) \in \mathbb{S},$$

with $i = 1, 2$. Similarly, by a similar argument for (4.3), we get

$$\begin{aligned} & \sup_x \mathbb{E} \left(\bar{v}_1(t,x) - \bar{v}_2(t,x) \right)^2 \\ \leq \ & C \int_0^t (t-s)^{-\frac{1}{2}} \sup_y \mathbb{E} \left(\bar{v}_1(s,y) - \bar{v}_2(s,y) \right)^2 ds \\ & + C \int_0^t (t-s)^{-\frac{1}{2}} \sup_y \mathbb{E} \left(\bar{u}_1(s,y) - \bar{u}_2(s,y) \right)^2 ds. \end{aligned} \tag{4.5}$$

Let

$$\Lambda(t) = \sup_x \mathbb{E}(\bar{u}_1(t,x) - \bar{u}_2(t,x))^2 + \sup_x \mathbb{E} \left(\bar{v}_1(t,x) - \bar{v}_2(t,x) \right)^2.$$

Jointing with (4.4) and (4.5), we get

$$\Lambda(t) \leq C \int_0^t \left(1 + (t-s)^{-\frac{1}{2}} \right) \Lambda(s) ds.$$

So

$$\Lambda(t) = 0, \quad as \quad t \in [0,T].$$

Then

$$(\bar{u}_1(t,x), \bar{v}_1(t,x)) = (\bar{u}_2(t,x), \bar{v}_2(t,x)), \quad as \quad (t,x) \in [0,T] \times D,$$

in L^2 sense. Then we get the result of this theorem.

□

4.3 Regularity

Let $(u(t,x), v(t,x))$ be the solution of the equation (4.2). Then $u(t,x) \in \mathbb{S}$ and $v(t,x) \in \mathbb{S}$, and $u(t,x)$ is the solution of the equation (2.3). By similar arguments as in the proof of the Hölder continuity of the solution pair $(u(t,x), u^\theta(t,x))$ to the equation (2.4) in Section 3.3, one can show the Hölder continuity of $u(t,x)$ and $v(t,x)$ which we state as the following theorem, its proof is omitted.

Theorem 4.3. *Assume that H_{h_1,h_2}, H_{u_0} and H_g hold. Let $u(t,x)$ be the solution of the equation (2.3). Then $u(t,x)$ is μ_1-Hölder continuous in t and ν_1-Hölder continuous in x, where $\mu_1 \in (0, \frac{1}{2})$ and $\nu_1 \in (0,1)$. Moreover, $v(t,x) = \frac{\partial}{\partial x}u(t,x)$ is μ_2-Hölder continuous in t and ν_2-Hölder continuous in x, where $\mu_2 \in (0, \min\{\frac{\kappa}{2}, \frac{2h_1+h_2-1}{3}\})$ and $\nu_2 \in (0, \min\{\kappa, \frac{2h_1+h_2-1}{2}\})$.*

5 Appendix

In this section, we review, for the convenience of the reader, a few elementary estimates about the Green function which are used in the paper. Recall that $p(t,x,y)$ is the fundamental solution of the heat operator $\frac{\partial}{\partial t} - \frac{1}{2}\Delta$ on $[0,\infty)$ subject to the Dirichlet boundary condition, given by the following explicit formula

$$p(t,x,y) = \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x+y)^2}{2t}} \right).$$

Lemma 5.1. *For $(t,x,y) \in [0,T] \times D \times D$, we have*

$$|p(t,x,y)| \le Ct^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4t}},$$

$$\left| \frac{\partial}{\partial x} p(t,x,y) \right| \le Ct^{-1} e^{-\frac{(x-y)^2}{4t}},$$

$$\left| \frac{\partial}{\partial t} p(t,x,y) \right| \le Ct^{-\frac{3}{2}} e^{-\frac{(x-y)^2}{4t}},$$

$$\left| \frac{\partial^2}{\partial x^2} p(t,x,y) \right| \le Ct^{-\frac{3}{2}} e^{-\frac{(x-y)^2}{4t}},$$

and

$$\left| \frac{\partial^2}{\partial x \partial t} p(t,x,y) \right| \le Ct^{-2} e^{-\frac{(x-y)^2}{4t}}.$$

Let us for example prove the second one, and the proofs for others are similar. Since

$$\frac{\partial}{\partial x} p(t,x,y) = \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{(x-y)^2}{2t}} \times \frac{y-x}{t} + e^{-\frac{(x+y)^2}{2t}} \times \frac{y+x}{t} \right).$$

Let $y - x = \xi\sqrt{t}$. Then

$$\begin{aligned}\left|\frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-y)^2}{2t}}\frac{y-x}{t}\right| &= \left|\frac{1}{\sqrt{2\pi t}}e^{-\frac{(x-y)^2}{4t}} \times e^{-\frac{\xi^2}{4}}\frac{\xi}{\sqrt{t}}\right| \\ &\leq Ct^{-1}e^{-\frac{(x-y)^2}{4t}}.\end{aligned}$$

Similarly

$$\begin{aligned}\frac{1}{\sqrt{2\pi t}}e^{-\frac{(x+y)^2}{2t}}\frac{y+x}{t} &\leq Ct^{-1}e^{-\frac{(x+y)^2}{4t}} \\ &\leq Ct^{-1}e^{-\frac{(x-y)^2}{4t}}\end{aligned}$$

and the proof is complete. □

Similarly, as in Bally et al. [2], we have the following result.

Lemma 5.2. *Let u_0 be a ω-Hölder continuous real function with $0 < \omega \leq 1$. Then*

$$\left|\int_D p(t,x,z)u_0(z)dz - \int_D p(s,y,z)u_0(z)dz\right| \leq C\left(|t-s|^{\frac{\omega}{2}} + |x-y|^{\omega}\right)$$

for any $s, t \in [0, T]$ and $x, y \in D = [0, \infty)$.

In fact, by the semigroup property of $p(t, x, y)$, we have

$$\begin{aligned}&\int_D p(t,x,z)u_0(z)dz - \int_D p(s,x,z)u_0(z)dz \\ = &\int_D\int_D p(s,x,y)p(t-s,y,z)u_0(z)dydz - \int_D p(s,x,y)u_0(y)dy \\ = &\int_D p(s,x,y)\left(\int_D p(t-s,y,z)(u_0(z)-u_0(y))dz\right)dy,\end{aligned}$$

so that

$$\begin{aligned}&\left|\int_D p(t,x,z)u_0(z)dz - \int_D p(s,x,z)u_0(z)dz\right| \\ \leq\ &C\int_D p(s,x,y)\left(\int_D p(t-s,y,z)|z-y|^{\omega}dz\right)dy \\ \leq\ &C\int_D p(s,x,y)|t-s|^{\frac{\omega}{2}}dy \\ =\ &C|t-s|^{\frac{\omega}{2}}.\end{aligned}$$

For simplicity, set

$$\varphi(t,x) = \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}.$$

Then

$$p(t,x,y) = \varphi(t,x-y) - \varphi(t,x+y).$$

If $y > x > 0$ and $\lambda = y - x$, then

$$
\begin{aligned}
&\left|\int_D p(t,x,z)u_0(z)dz - \int_D p(t,y,z)u_0(z)dz\right| \\
= &\left|\int_D (\varphi(t,z-x) - \varphi(t,z-y))u_0(z)dz \right.\\
&\left. - \int_D (\varphi(t,z+x) - \varphi(t,z+y))u_0(z)dz\right| \\
= &\left|\int_D \varphi(t,z-x)(u_0(z) - u_0(z+\lambda))dz + \int_{-\lambda}^0 \varphi(t,z-x)u_0(z+\lambda)dz \right.\\
&\left. - \int_\lambda^{+\infty} \varphi(t,z+x)(u_0(z) - u_0(z-\lambda))dz - \int_0^\lambda \varphi(t,z+x)u_0(z)dz\right| \\
\le &\left|\int_D \varphi(t,z-x)(u_0(z) - u_0(z+\lambda))dz\right| \\
&+ \left|\int_\lambda^{+\infty} \varphi(t,z+x)(u_0(z) - u_0(z-\lambda))dz\right| \\
&+ \left|\int_0^\lambda \varphi(t,z+x)u_0(z)dz - \int_{-\lambda}^0 \varphi(t,z-x)u_0(z+\lambda)dz\right| \\
\le &\ C\lambda^\omega \int_D p(t,x,z)dz + \left|\int_\lambda^0 \varphi(t,z+x)(u_0(z) - u_0(\lambda - z))dz\right| \\
\le &\ C\lambda^\omega + C\int_0^\lambda \varphi(t,z+x)|2z-\lambda|^\omega dz \\
\le &\ C\lambda^\omega = C|x-y|^\omega.
\end{aligned}
$$

This completes the proof of the lemma.

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